A spectral element agglomeration AMG solver using element topology and element matrices

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We implement and study an element-based AMG solver which utilizes element agglomeration and spectral way of selecting coarse degrees of freedom. Element-based AMG methods target linear systems coming from finite element discretizations of elliptic PDE problems. Such systems often have non-M-matrices and therefore may not be suitable for more classical AMG approaches. Our solver uses individual element information (element matrices, element topology). This information is recursively built on coarser levels by the solver. The selection of the coarse dofs is done by solving large number of small eigenvalue problems, constructing of tentative prolongator from the computed eigenvectors and applying few special smoothing steps to tentative prolongator.

1 Introduction

Our goal is to solve a linear system

$$Au = f$$

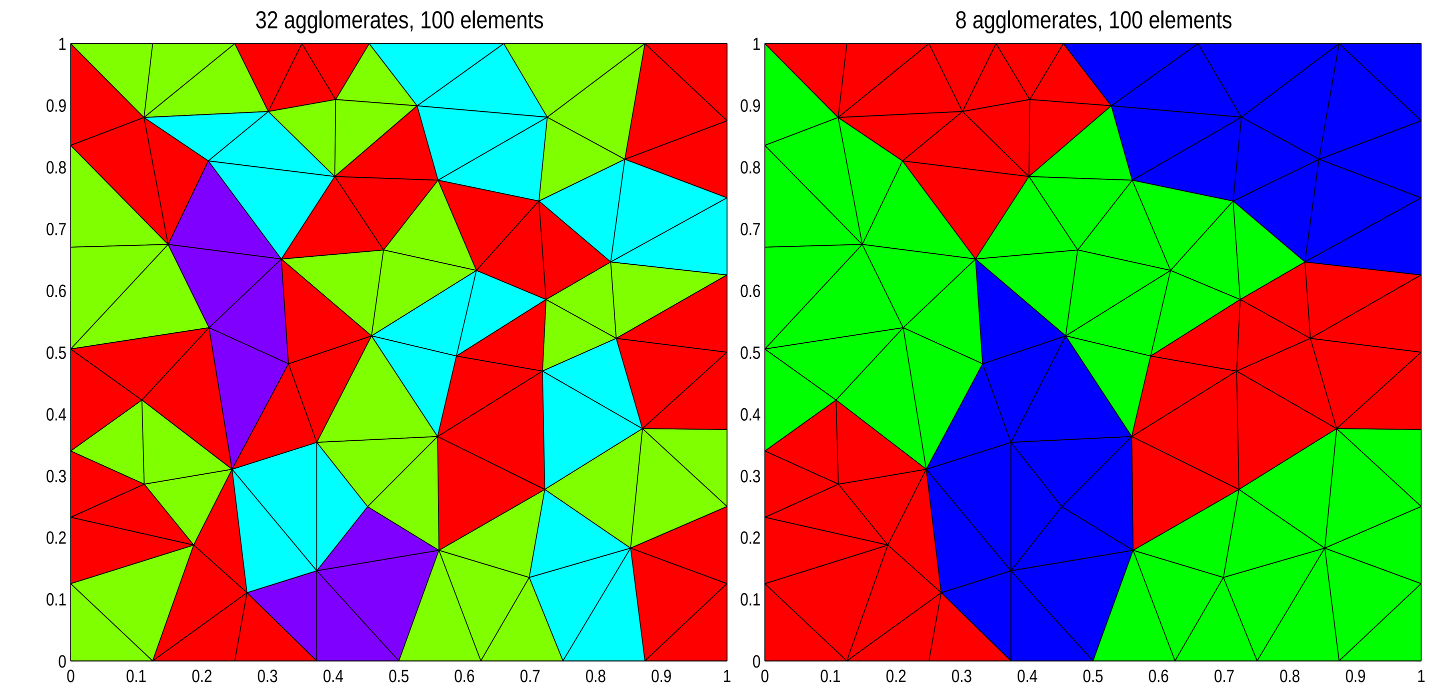
where A is symmetric positive definite matrix coming from finite element discretization. We are given the following information:

- Individual element matrices.
- Topological relations (which dofs belong to which elements and what elements are neighbours, i.e. share a face).

We need to create tools necessary for multigrid V-cycles (coarse-grid matrices and prolongators).

2 Agglomerating elements

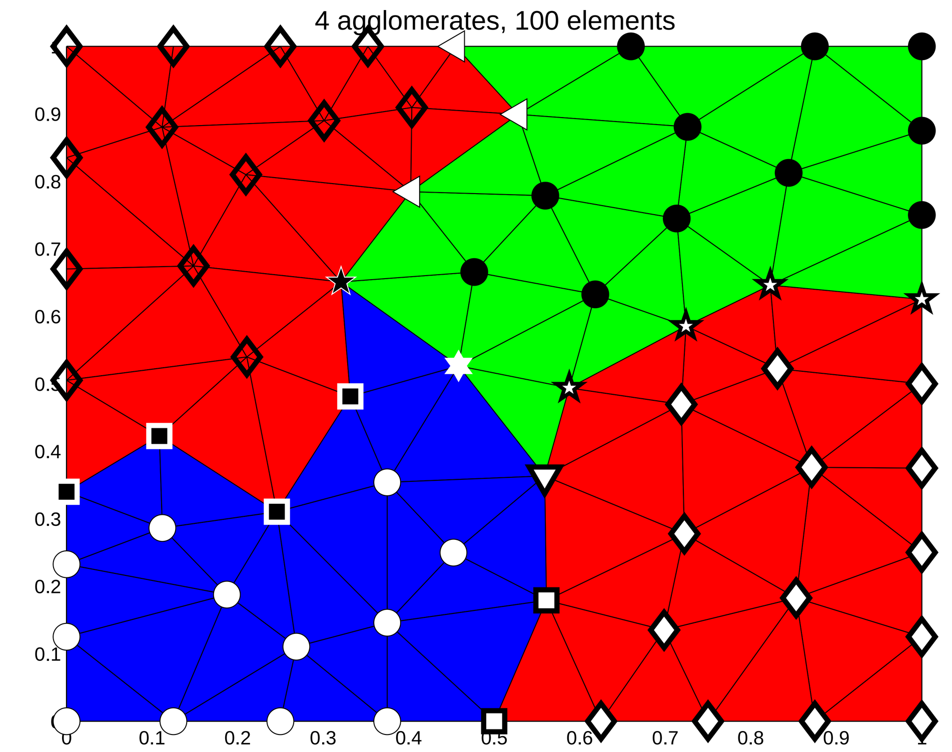
We start with splitting the set of all (fine) elements into non-intersecting subsets, called *agglomerates* or *agglomerated elements (AEs)*. More precisely, we treat elements as vertices in a non-oriented graph, where 2 vertices (elements) are linked with an edge, if and only if they are neighbours, i.e. share a face. We apply METIS software to partition this graph into desired number of components. Each component represents an agglomerate. We make sure that produced components are connected.



Agglomerated elements

3 Minimal intersection sets

Having created agglomerated elements, we can split the set of all (fine) degrees of freedom into non-overlapping partitions, called *minimal intersection sets*. We place 2 dofs in the same minimal intersection set if and only if these 2 dofs belong to the same set of agglomerated elements. For example, interior dofs of an AE constitute a minimal intersection set. A boundary between 2 AEs constitutes a minimal intersection set and so on.



Minimal intersection sets

4 Solving eigenproblems

For each minimal intersection set I we define its neighbourhood $N(I)$ which consists of all (fine) elements sharing a dof from I. We assemble the local matrix $A_{N(I)}$ from these elements and then calculate its Schur complement S_I by eliminating the dofs outside I. Then we solve (small and dense) eigenvalue problem for each S_I and peek 1 or several lower eigenmodes. We extend the selected eigenvectors by zero outside their domain (I) and use them as columns of our tentative prolongator. It can be proved, that, if one chooses sufficient number of eigenmodes of each Schur complement, the restriction of the globally assembled matrix corresponding to the orthogonal complement of the range of our tentative prolongator will be well-conditioned.

5 Building coarse element matrices

To be able to apply our algorithm recursively, we need to construct coarse elements, coarse dofs and coarse element matrices. We use agglomerated elements as coarse elements, and we use coefficients in the basis of eigenmodes of Schur complements as coarse dofs. Thus, each coarse element can be split into several minimal intersection sets and each minimal intersection set has one or several coarse dofs (eigenmodes) associated with it. This allows to associate coarse elements with coarse dofs. To construct coarse element matrices, we first assemble local matrices of agglomerated elements, then, for each agglomerated element E, we take a submatrix $\bar{P}_E$ of tentative prolongator $\bar{P}$, corresponding to coarse and fine dofs associated with E. After that, we construct coarse element matrix A^c_E using the formula $A^c_E = \bar{P}^T_E A_E \bar{P}_E$, where A_E is local assembled matrix of agglomerated element.

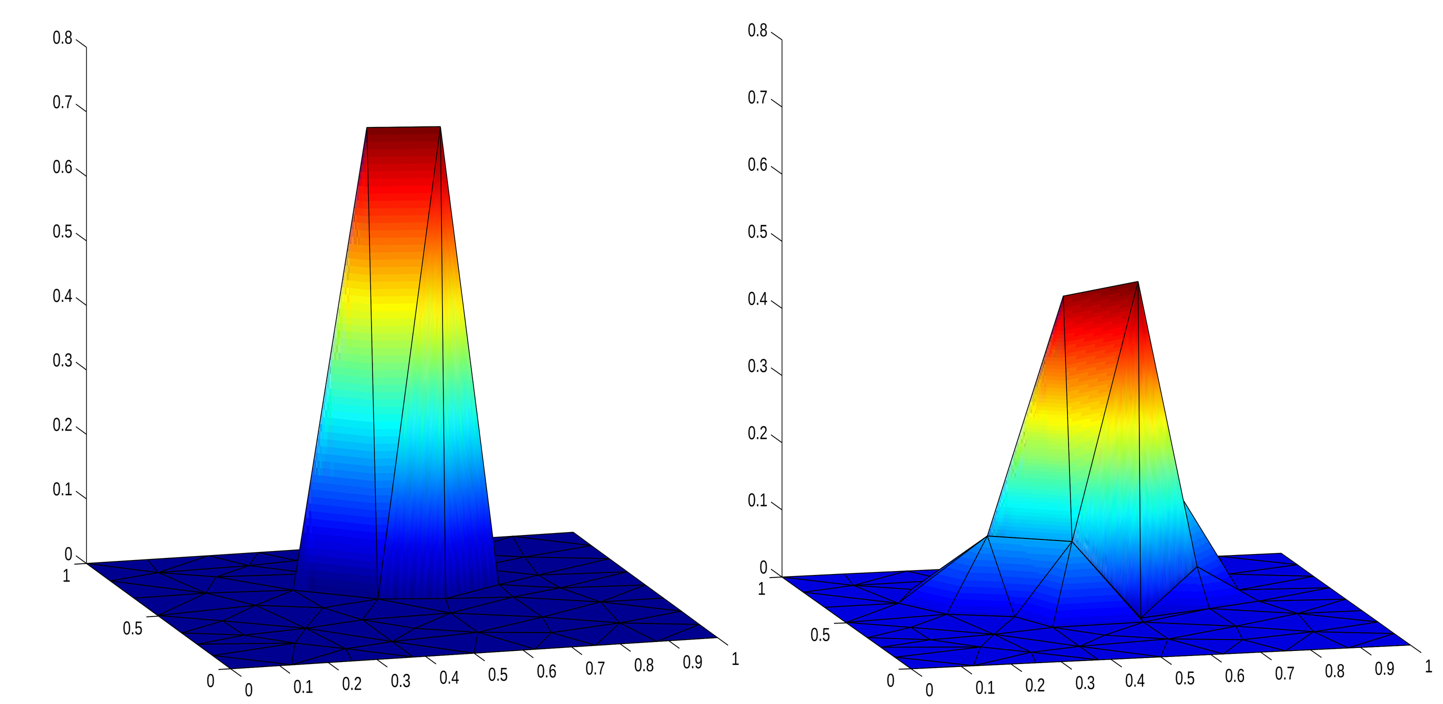
6 Smoothing tentative prolongator

To improve the prolongator quality, we apply one or several special smoothing steps to tentative prolongator $\bar{P}$. We use the following formula:

$$P = \left(I - D^{-1}A\right)\bar{P}$$

where D is a diagonal matrix with entries

$$D_{ii} = \sum_{j=1}^{n} |A_{ij}|$$



Tentative and smoothed prolongator (1 column)

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